Almost dual pairs and definable classes of modules

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Abstract

In [7] Holm considers categories of right modules dual to those with support in a set of finitely presented modules. We extend some of his results by placing them in the context of elementary duality on definable subcategories.

1 Introduction

Let $\mathcal{B} = \mathrm{add}(\mathcal{B})$ be an additive subcategory of R-mod, the category of finitely presented left R-modules. Lenzing [15] studied properties of those categories of the form $\varinjlim \mathcal{B}$, where this denotes the closure of $\mathcal{B}$ under direct limits in the category, R-Mod, of all left R-modules. Given a left module M, denote by M^* its dual (right R-) module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. Holm ([7]) considers the closure, $\mathrm{Prod}(\mathcal{B}^*)$, of $\mathcal{B}^*$ under direct products and direct summands in Mod-R, which he refers to as the category of modules with cosupport in $\mathcal{B}$ (his notation, which we will not use, for this is $(\text{Mod-}R)^{\mathcal{B}}$). In this paper we set the duality between categories such as $\varinjlim \mathcal{B}$ and $\mathrm{Prod}(\mathcal{B}^*)$ in a more general context and we extend some of the results from Holm's paper.

Every dual of a module is pure-injective and the classes of pure-injectives which are closed under products and direct summands correspond bijectively (by taking their closures under pure submodules) to the type-definable classes of modules considered by Burke ([2]) and hence also to the closed subsets in the full support topology that he defined on the set of indecomposable pure-injective modules. These type-definable classes were introduced as an extension of the definable classes which arose in the model theory of modules ([28], [19]). From

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this perspective it is natural to extend results from [7] by considering the closure of classes under pure submodules. There is a duality, elementary duality, between definable classes ([6]) and, again, this provides a perspective which allows us to clarify the relation between classes such as $\varinjlim \mathcal{B}$ and $\mathrm{Prod}(\mathcal{B}^*)$.

For a class $\mathcal{X}$ of modules (we always assume our classes to be closed under isomorphism) we write $\mathrm{add}(\mathcal{X})$ (respectively $\mathrm{Add}(\mathcal{X})$) for the closure of $\mathcal{X}$ under finite (resp. arbitrary) direct sums and direct summands, we set $\mathcal{X}^* = \{M^* : M \in \mathcal{X}\}$ to be the class of duals of modules in $\mathcal{X}$, we denote by $\mathrm{Prod}(\mathcal{X})$ the closure of $\mathcal{X}$ under direct products and direct summands and by $\mathrm{Prod}^+(\mathcal{X})$ we denote the closure of $\mathcal{X}$ under direct products and pure submodules. We also write $\mathcal{X}^+$ for the closure of $\mathcal{X}$ under pure submodules. We write $\mathrm{Pinj}(\mathcal{X})$ for the class of pure-injective modules which are in $\mathcal{X}$. By pinj_R we denote the set of isomorphism classes of indecomposable pure-injective right R-modules. We will use [20] as a handy reference for definitions and results around purity; there are many other sources.

Recall, e.g. [20, 4.3.29], than any character/dual module M^* is pure-injective.

Let $\mathcal{S}$, respectively $\mathcal{P}$, denote subclasses (or subcategories) of R-Mod, respectively Mod-R. We say that $(\mathcal{S}, \mathcal{P})$ is an **almost dual pair** if:

1. $\mathcal{P} = \mathrm{Prod}(\mathcal{P})$ and $\mathcal{P}$ is closed under pure-injective hulls[1]
2. $M \in \mathcal{S}$ iff $M^* \in \mathcal{P}$.

Immediate examples include: $(R\text{-Mod}, \mathrm{Pinj}(\text{Mod-}R))$; $(R\text{-Flat}, \text{Abs-}R)$, $(R\text{-Flat}, \text{Inj-}R)$ ([14, p. 239]) where we use obvious notation for flat, absolutely pure (=fp-injective) and injective modules; the pair $(R\text{-Abs}, \text{Flat-}R)$ is almost dual iff R is left coherent ([26, 1.6]).

Lemma 1.1. *Suppose that $(\mathcal{S}, \mathcal{P})$ is an almost dual pair. If $0 \to L \to M \to N \to 0$ is a pure-exact sequence, then $M \in \mathcal{S}$ iff $L, N \in \mathcal{S}$. Moreover $\mathcal{S} = \mathrm{Add}(\mathcal{S}) = \varinjlim \mathcal{S}$.*

Proof. If $0 \to L \to M \to N \to 0$ is a pure-exact sequence with $M \in \mathcal{S}$ then, applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$, we obtain the split exact sequence (e.g. [20, 4.3.30]) $0 \to N^* \to M^* \to L^* \to 0$, from which we see that $M^* \in \mathcal{P}$ iff $L^*, N^* \in \mathcal{P}$, whence we obtain the first statement.

Also, since $(\bigoplus_{i\in I} M_i)^* = \prod_{i\in I} M_i^*$, the first equality of the second statement is immediate from the definitions and since, for any directed system $(M_i)_i$ the canonical map $\bigoplus_i M_i \to \varinjlim_i M_i$ is a pure epimorphism ([23, p. 56]), the second equality follows from the first assertion. □

Corollary 1.2. *If $(\mathcal{S}, \mathcal{P})$ is an almost dual pair then $\mathcal{S}$ is (pre-)covering in R-Mod.*

This follows directly by [9, 2.5], [12, Thm. 4].

Although $\mathcal{S}$ is completely determined by $\mathcal{P}$ the converse is not true: if $(\mathcal{S}, \mathcal{P})$ is an almost dual pair then both $(\mathcal{S}, \mathrm{Pinj}(\mathcal{P}))$ and $(\mathcal{S}, \mathcal{P}^+)$ are

[1] In [17], and also in Holm and Jørgensen's notion of a duality pair [9], the class $\mathcal{P}$ is not required to be closed under pure-injective hulls.

almost dual pairs and these are equal iff $\mathcal{P}^+$ consists only of pure-injectives, but that is a strong condition (which we consider in Section 5), being equivalent to Σ-pure-injectivity of every member of $\mathcal{P}$. We will see that an additional condition is needed for these to be the upper and lower bounds of the possibilities for $\mathcal{P}$.

Indeed, if $(\mathcal{S}, \mathcal{P})$ is an almost dual pair and if $\mathrm{Prod}(\mathcal{S}^*) = \mathrm{Pinj}(\mathcal{P})$ then we will say that this is a **dual pair**[2]; in this case it follows directly that $\mathrm{Pinj}(\mathcal{S}^*) \subseteq \mathcal{P} \subseteq (\mathcal{S}^*)^+$. All the examples above are actually dual pairs but we will give an example later (3.4) of an almost dual pair with $\mathrm{Prod}(\mathcal{S}^*)$ properly contained in $\mathrm{Pinj}(\mathcal{P})$. In order to establish that example, we need to distinguish between arbitrary classes of pure-injectives closed under products and direct summands and those which arise by closing classes of duals of modules under these operations; we do this in Section 3. In that section we also show that the definition of almost dual pair is independent of the choice of duality - for example, if R is an algebra over a field K then it would be natural to use $\mathrm{Hom}_K(-, K)$ in place of $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$, or one might use the "local" duality $M \mapsto \mathrm{Hom}_S(M, E)$ where $S = \mathrm{End}(M)$ and E is an injective cogenerator for S-modules.

First, however, we note the connection with torsion theories on the associated functor category and with topologies on pinj_R.

2 Type-definable subcategories and torsion theories on the functor category

A class $\mathcal{X}$ of (right R-)modules is said to be **type-definable** ([2], see [20, §§5.3.7, 12.7]) if it is closed under pure-injective hulls, direct products and pure submodules (the actual definition is in terms of pp-types but this is equivalent). If a class $\mathcal{X}$ is type-definable then $\mathrm{Pinj}(\mathcal{X})$ is a class $\mathcal{P}$ of pure-injectives satisfying $\mathcal{P} = \mathrm{Prod}(\mathcal{P})$ and every such class of pure-injectives arises in this way from a type-definable class. There is a natural bijection between these classes and hereditary torsion theories on the (locally coherent, Grothendieck abelian) functor category $(R\text{-mod}, \mathbf{Ab})$.

This bijection is induced by the full embedding $\epsilon : \text{Mod-}R \to (R\text{-mod}, \mathbf{Ab})$ which takes M_R to the functor $(M \otimes_R -)$ and which has the natural action on morphisms. The image under ϵ of an exact sequence is exact iff the original sequence is pure-exact and the image of a module is injective iff the original module is pure-injective (see [5, §1], [10, B16] or [20, §12.1]). Hereditary torsion theories on Grothendieck abelian categories are in bijection with classes $\mathcal{I}$ of injective objects closed under direct products and direct summands - the torsionfree class being the class of subobjects of objects in $\mathcal{I}$ - so we have the following.

Remark 2.1. There are natural bijections (as described above) between:

[2]Note that this is considerably more restrictive than the similarly-named notion of duality pair from [9].

type-definable classes of right R-modules;
classes $\mathcal{P}$ of pure-injective right R-modules satisfying $\mathcal{P} = \mathrm{Prod}(\mathcal{P})$
hereditary torsion theories on (R-mod, $\mathbf{Ab}$).

We will see (3.4) that not every such class $\mathcal{P} = \mathrm{Prod}(\mathcal{P})$ of pure-injectives has the form $\mathrm{Prod}(\mathcal{S}^*)$ for some class $\mathcal{S}$ of modules.

Remark 2.2. There are natural bijections between the following:
classes $\mathcal{P}$ of pure-injective right R-modules of the form $\mathrm{Prod}(\mathcal{S}^*)$ for some class $\mathcal{S}$ of left R-modules;
dual pairs $(\mathcal{S}, \mathcal{P})$ with minimal $\mathcal{P} = \mathrm{Prod}(\mathcal{S}^*)$, that is, $\mathcal{P} = \mathrm{Pinj}(\mathcal{P})$;
dual pairs $(\mathcal{S}, \mathcal{P})$ with maximal $\mathcal{P} = \mathrm{Prod}(\mathcal{S}^*)^+$, that is $\mathcal{P} = \mathcal{P}^+$;

A subcategory of a module category is a **definable subcategory** if it is closed under direct products, direct limits and pure submodules, equivalently if it is type-definable and closed under direct limits. In the bijection above these correspond exactly to the torsion theories of finite type (see [20, 12.4.1]), meaning that the torsion class is generated by finitely presented objects.

The results in this paper could be presented, as Holm does to some extent in [7], in a way which makes use of this functor-category perspective.

3 Dualities

If M is a left R-module, $S \to \mathrm{End}(M)$ is a ring homomorphism and E is an injective cogenerator for right S-modules then we will say that $\mathrm{Hom}_S(-, E_S)$ is a duality that applies to M and we will write M^* for the right R-module $\mathrm{Hom}_S(M_S, E_S)$. In this section we will use the notions of pp formula and pp-type and associated results from the model theory of modules ([20] is one reference for these) because these apply nicely to the relation between M and M^*. In particular we need the following, for which see [29, §2(c)], [22, 1.5] (or [20, 1.3.12]).

Proposition 3.1. *Let M be any left R-module, let $\phi(v)$ be a pp formula (with one free variable) for right R-modules and let * be a duality that applies to M. Then the solution set, $\phi(M^*)$, of ϕ in M^* is the annihilator $\{f \in M^* : f \cdot D\phi(M) = 0\}$ of the solution set of $D\phi$ in M. That is, $f \in \phi(M^*)$ iff $D\phi(M) \leq \ker(f)$.*

Here $D\phi$ is the elementary dual of the formula ϕ ([18], see [20, §1.3]). Duality applied twice is equivalent to the identity - $DD\phi(M) = \phi(M)$ for every module M - so the result with the roles of ϕ and $D\phi$ reversed also is true. We freely use the fact that for every module M and pp formula ϕ, it is the case that $\phi(M)$ is an $\mathrm{End}(M)$-submodule of M (see [20, 1.1.8]).

Theorem 3.2. *Suppose that * and $^\sharp$ are dualities each of which applies to the module M. Then $\mathrm{Prod}(M^*) = \mathrm{Prod}(M^\sharp)$.*

Proof. It is enough to show that $M^* \in \mathrm{Prod}(M^\sharp)$. To establish that, it will enough to prove that for each nonzero $f \in M^*$ there is $g \in (M^\sharp)^I$ for some set I, such that the **pp-type of f in M^*** - the set of all pp

formulas ϕ such that $f \in \phi(M^*)$ - equals the pp-type of g in $(M^\sharp)^I$. For then there will, by [20, 4.3.9], be a morphism α_f from M^* to $(M^\sharp)^I$ taking f to g. The product over all $f \in M^*$ of these maps α_f will then be a pure, hence split, embedding of M^* into a direct product of copies of $M^\sharp$. Let us write p for the pp-type of f in M^*.

By 3.1, for each $\phi \in p$, $D\phi(M) \leq \ker(f)$, therefore $\sum_{\phi \in p} D\phi(M) \leq \ker(f)$, and for each pp formula (in the same free variable v) ψ not in p (let us write p^- for the set of these), $D\psi(M) \not\leq \ker(f)$, in particular $D\psi(M) \not\leq \sum_{\phi \in p} D\phi(M)$.

Therefore since these are S-modules and E is an injective cogenerator for S-modules (with the notation introduced above), there is $g_\psi \in M^\sharp$ such that $\sum_{\phi \in p} D\phi(M) \leq \ker(g_\psi)$ and $D\psi(M) \not\leq \ker g_\psi$. So, by 3.1, for all $\phi \in p$, $g_\psi \in \phi(M^\sharp)$ but $g_\psi \notin \psi(M^\sharp)$.

Set $g = (g_\psi)_{\psi \in p^-} \in (M^\sharp)^{p^-}$. Then, since pp formulas commute with direct products ([20, 1.2.3]), the pp-type of g is precisely p, as required. $\square$

Theorem 3.3. *Let M be any nonzero module and let M^* be a dual of M. Then M^* has an indecomposable direct summand.*

Proof. Choose $a \in M$, $a \neq 0$. By Zorn's Lemma there is, in the lattice of pp-definable subgroups of M, a lattice ideal $\mathcal{I}$ such that, for all $\psi(M) \in \mathcal{I}$, $a \notin \psi(M)$ and $\mathcal{I}$ is maximal such. For notational simplicity, let us write $\psi \in \mathcal{I}$ rather than $\psi(M) \in \mathcal{I}$. If ϕ is such that $\phi(M) \notin \mathcal{I}$ then, by maximality of $\mathcal{I}$, $a \in \psi(M) + \phi(M)$ for some $\psi \in \mathcal{I}$.

Take $f \in M^*$ such that $\sum_{\psi \in \mathcal{I}} \psi(M) \leq \ker(f)$ and $f(a) \neq 0$. By 3.1 (with dual formulas on the other side), $f \in D\psi(M^*)$ for all $\psi \in \mathcal{I}$ but, for all $\phi \notin \mathcal{I}$, $f \notin D\phi(M^*)$: for, given $\phi \notin \mathcal{I}$, choose $\psi \in \mathcal{I}$ such that $a \in \psi(M) + \phi(M) = (\psi + \phi)(M)$ so, by 3.1, $f \notin D(\psi + \phi)(M^*) = (D\psi(M^*) \cap D\phi(M^*))$ so, since $f \in D\psi(M^*)$, $f \notin D\phi(M^*)$ as claimed. Therefore the pp-type, p, of f in M^* is $\{D\psi : \psi \in \mathcal{I}\}$.

We claim that this pp-type, p, is **irreducible** (meaning that it is the pp-type of some element in an indecomposable pure-injective). We check Ziegler's criterion ([28, 4.4], see [20, 4.3.49]). For that, we take any pp formulas $D\phi_1, D\phi_2 \notin p$ that is, as shown above, with $\phi_1, \phi_2 \notin \mathcal{I}$. By maximality of $\mathcal{I}$ there are $\psi_1, \psi_2 \in \mathcal{I}$ such that $a \in (\psi_i + \phi_i)(M)$ $(i = 1, 2)$ so, setting $\psi = \psi_1 + \psi_2$, we have $a \in (\psi + \phi_1)(M) \cap (\psi + \phi_2)(M)$. Since $f(a) \neq 0$, 3.1 gives $f \notin D((\psi + \phi_1) \wedge (\psi + \phi_2))(M^*)$. That is, $f \notin ((D\psi \wedge D\phi_1) + (D\psi \wedge D\phi_2))(M^*)$. That is, we have $D\psi \in p$ such that $(D\psi \wedge D\phi_1) + (D\psi \wedge D\phi_2) \notin p$, and that is Ziegler's criterion, so our claim is established.

Therefore any element with pp-type p in a pure-injective module is contained in an indecomposable summand of that module (see [20, §4.3.5, esp. 4.3.46]); applied to $f \in M^*$, we have the theorem. $\square$

Here is our example.

Example 3.4. Let R be the free associative algebra $K\langle X, Y\rangle$ over a field K. Then R is a domain with no uniform one-sided ideal, so its injective hull E has no indecomposable direct summand. Nor does

any product of copies of E have a direct summand, since any nonzero submodule of a product of copies of E must embed a copy of R. It follows by 3.3 that $(0, \mathrm{Prod}(E))$ is an almost dual pair, with the first class in no sense determining the second.

This shows that not every almost dual pair has the form $(\mathcal{S}, \mathcal{P})$ with $\mathrm{Prod}^+(\mathcal{S}^*) \subseteq \mathcal{P} \subseteq \mathrm{Prod}(\mathcal{S}^*)$, in particular not all classes $\mathcal{P}$ as in 2.1 arise from dual pairs, equivalently not all torsion theories on the functor category arise this way. In particular, from 3.5 below we see that the torsion theories which arise from dual pairs of the form $(\mathcal{S}, \mathrm{Prod}(\mathcal{S}^*))$ are cogenerated by indecomposable injectives.

If $\mathcal{D}$ is a definable subcategory of Mod-R and $N \in \mathrm{Pinj}(\mathcal{D})$ is indecomposable then N is **neg-isolated** in $\mathcal{D}$ if it is the hull of a pp-type which is neg-isolated modulo $\mathcal{D}$, equivalently if, whenever N is a direct summand of a product $\prod_\lambda N_\lambda$ in $\mathrm{Pinj}(\mathcal{D})$ already N is a direct summand of some N_λ (see [20, §5.3.5, esp. 5.3.48]); another equivalent is that $(N \otimes -)$ is the injective hull of a functor in (R-mod, $\mathbf{Ab}$) which becomes a simple object in the (finite type) localisation of that functor category at the torsion theory cogenerated by the $(N' \otimes -)$ with $N' \in \mathrm{Pinj}(\mathcal{D})$ ([20, 12.5.6]).

Proposition 3.5. *Let M be any nonzero module and let M^* be a dual of M. Then* $\mathrm{Prod}(M^*)$ *is cogenerated by indecomposable pure-injectives, indeed* $\mathrm{Prod}(M^*) = \mathrm{Prod}(\mathcal{N})$ *where $\mathcal{N}$ is the set of direct summands of M^* which are neg-isolated in the definable category generated by M^*.*

Proof. We get this by combining the arguments of 3.2 and 3.3. Take $f \in M^*$, $f \neq 0$ and let p be the pp-type of f in M^*. As before, for each $\psi \in p^-$ we have $D\psi(M) \nleq \sum_{\phi \in p} D\phi(M)$. Choose $a_\psi \in D\psi(M) \setminus \sum_{\phi \in p} D\phi(M)$. As in the proof of 3.3 choose a lattice ideal $\mathcal{I}$ in $\mathrm{pp}(M)$ which contains all the $D\phi(M)$ with $\phi \in p$ and such that $a \notin \sum_{D\phi \in \mathcal{I}} D\phi(M)$. Then there is $f_\psi \in M^*$ such that $f_\psi(\sum_{D\phi \in \mathcal{I}} D\phi(M)) = 0$ - hence such that $f_\psi \in \phi(M^*)$ for each $D\phi \in \mathcal{I}$ - and such that $f(a) \neq 0$ so, by 3.1, $f \notin D\psi(M^*)$. As in the proof of 3.3 the pp-type of f_ψ is irreducible, indeed neg-isolated in the theory of M^* by ψ (this means that $\mathcal{I}$ is determined uniquely as a lattice ideal by the fact that it contains $\mathcal{I}$ and does not contain ψ). This means that if we choose any minimal direct summand N_ψ of M^* which contains f_ψ then this is indecomposable and neg-isolated in the definable subcategory generated by M^*. Then, just as in the proof of 3.2, we deduce that M^* embeds into a product of such neg-isolated pure-injectives, which is enough. $\square$

4 Definable subcategories

There is a natural bijection, elementary duality ([6, 6.6], see [20, §3.4.2]) between the definable subcategories of R-Mod and those of Mod-R which can be defined in various ways, the most natural here being to take a definable subcategory $\mathcal{D}$ of R-Mod to $\mathcal{D}^{\mathrm{d}} = (\mathcal{D}^*)^+$, which is a

definable subcategory of Mod-R (see [20, 1.3.15])) and, see [20, 3.4.21], $(\mathcal{D}^{\mathrm{d}})^{\mathrm{d}} = \mathcal{D}$. The next observation, therefore is immediate.

Remark 4.1. If $\mathcal{D}$ is a definable category of right R-modules and $\mathcal{D}^{\mathrm{d}}$ denotes the elementary dual definable category of left R-modules then both $(\mathcal{D}^{\mathrm{d}}, \mathcal{D})$ and $(\mathcal{D}, \mathcal{D}^{\mathrm{d}})$ are dual pairs. We will refer to any such pair as a dual pair of definable categories.

Here is an example of a dual pair which is not definable.

Example 4.2. Consider the almost dual pair $(\mathrm{Add}(\mathbb{Z}_{p^\infty}), \mathrm{Prod}(\overline{\mathbb{Z}_{(p)}}))$ which is "cogenerated" by the p-adic integers $\overline{\mathbb{Z}_{(p)}}$ regarded as a $\mathbb{Z}$-module. This is a dual pair, since $(\mathrm{Add}(\mathbb{Z}_{p^\infty}))^* = \mathrm{Prod}(\overline{\mathbb{Z}_{(p)}})$, and can equally be regarded as being "generated" by the Prüfer group $\mathbb{Z}_{p^\infty}$.

The dual of $\overline{\mathbb{Z}_{(p)}}$ is $\mathbb{Z}_{p^\infty} \oplus \mathbb{Q}^{(2^{\aleph_0})}$, which is not in $\mathrm{Add}(\mathbb{Z}_{p^\infty})$ so this is not a dual pair of definable subcategories (cf. 4.4). Indeed the dual pair consisting of definable subcategories which minimally contains this pair is $(\mathrm{Add}(\mathbb{Z}_{p^\infty} \oplus \mathbb{Q}), \mathrm{Prod}^+(\overline{\mathbb{Z}_{(p)}} \oplus \mathbb{Q}))$.

Proposition 4.3. *([15, 2.2], [3, 4.2], see [7, 4.1]) Suppose that $(\mathcal{S}, \mathcal{P})$ is an almost dual pair over R. Then the following are equivalent:*
(i) $\mathcal{S}$ is definable;
(ii) $\mathcal{S}$ is closed under products;
*(iii) $\mathcal{S}$ is preenveloping in R-*Mod.

Example 3.4 shows that if $(\mathcal{S}, \mathcal{P})$ is an almost dual pair and $\mathcal{S}$ is definable then it need not be that $\mathcal{P}$ is definable (that is, closed under direct limits, equivalently under pure epimorphisms). On the other hand definability of $\mathcal{P}$ (which implies $\mathcal{P}^+ = \mathcal{P}$) does imply definability of $\mathcal{S}$.

Theorem 4.4. *Suppose that $(\mathcal{S}, \mathcal{P})$ is an almost dual pair over R. Then the following are equivalent:*
(i) $\mathcal{P}^+$ is definable;
(ii) $\mathcal{P}^* \subseteq \mathcal{S}$;
(iii) $(\mathcal{P}^+)^* \subseteq \mathcal{S}$
(iv) $\mathcal{S}$ is definable and $\mathrm{Pinj}(\mathcal{P}) \subseteq \mathrm{Prod}(\mathcal{S}^*)$*;*
(v) $\mathcal{S}$ is definable and every $A \in \mathcal{P}^+$ is pure in the dual of some module from $\mathcal{S}$;
(vi) $\mathcal{S}$ is definable and $(\mathcal{S}, \mathcal{P})$ is a dual pair;
(vii) $(\mathcal{S}, \mathcal{P}^+)$ (and hence also $(\mathcal{P}^+, \mathcal{S})$) is a dual pair of definable subcategories.

Proof. (i)⇒(iii) Let $A \in \mathcal{P}^+$; since $\mathcal{P}^+$ is definable, $A^{**} \in \mathcal{P}^+$ so, being both pure-injective and pure in some member of $\mathcal{P}$, A^{**} is in $\mathcal{P}$. Therefore, by the definition of almost dual pair, $A^* \in \mathcal{S}$.

Clearly (iii)⇒(ii).

(ii)⇒(iv) The second condition follows from the fact that any pure-injective is a direct summand of its double-dual. To show that $\mathcal{S}$ is definable it will be enough, by 4.3, to show that $\mathcal{S}$ is closed under direct products, so take $A_i \in \mathcal{S}$, $i \in I$. Then for each i, $A_i^* \in \mathcal{P}$ so $\prod_i A_i^* \in \mathcal{P}$. Since the canonical embedding $\bigoplus_i A_i^* \to \prod_i A_i^*$ is pure, the dual map gives $(\bigoplus_i A_i^*)^*$ as a direct summand of $(\prod_i A_i^*)^*$ which,

by assumption, is in $\mathcal{S}$. Also by assumption each A_i^{**} is in $\mathcal{S}$. So $\prod_i A_i$, which is pure in $\prod_i A_i^{**} \simeq (\bigoplus_i A_i^*)^*$, is, by 1.1, in $\mathcal{S}$, as required.

(iv)⇒(i) Let $M \in \mathcal{P}^+$, say M is pure in $N \in \mathrm{Pinj}(\mathcal{P})$. By assumption there is $B \in \mathcal{S}$ with N a direct summand of B^*, so M purely embeds in B^* and, dualising, M^* is a direct summand of B^{**} which, since $\mathcal{S}$ is definable, is in $\mathcal{S}$. Hence $M^* \in \mathcal{S}$ and we have $(\mathcal{P}^+)^* \subseteq \mathcal{S}$.

Suppose that we have a directed system $(A_i)_{i\in I}$ in $\mathcal{P}^+$ with direct limit A. As in 1.1, A is a pure epimorphic image of the direct sum of the A_i and, dualising, we obtain a split embedding $A^* \to \left(\bigoplus_i A_i\right)^* = \prod_i A_i^*$. We have just seen that each A_i^* is in $\mathcal{S}$, hence $A^* \in \mathcal{S}$ and therefore $A^{**} \in \mathcal{P}$. Therefore $A \in \mathcal{P}^+$, as required.

(iv)⇔(v) This is immediate.

(iv)(⇒)(vi) is immediate from the definitions, as is (i)+(iv)(⇒)(vii). Both (vi)(⇒)(iv) and (vii)(⇒)(i) are immediate. $\square$

Example 4.2 shows that definability of $\mathcal{S}$ cannot be dropped from condition (iv).

Corollary 4.5. *If $(\mathcal{S}, \mathcal{P})$ is an almost dual pair of R-modules and $\mathcal{P}$ is a definable subcategory of* Mod-R *then $(\mathcal{P}, \mathcal{S})$ also is a(n almost) dual pair of R-modules, and $\mathcal{S} = (\mathcal{P}^*)^+$.*

Corollary 4.6. *If $\mathcal{S}$ is a definable subcategory of R-*Mod *then both $(\mathcal{S}, \mathrm{Prod}^+(\mathcal{S}^*))$ and $(\mathrm{Prod}^+(\mathcal{S}^*), \mathcal{S})$ are dual pairs of definable categories.*

5 Almost dual pairs generated by finitely presented modules

Holm showed that if $\mathcal{B}$ is an additive subcategory of R-mod then $(\varinjlim \mathcal{B}, \mathrm{Prod}(\mathcal{B}^*))$ is a(n almost) dual pair. We will give a somewhat modified proof of this here.

Theorem 5.1. *([7, 1.4]) Let $\mathcal{B}$ be an additive subcategory of R-*mod*. Then $M \in \varinjlim \mathcal{B}$ iff $M^* \in \mathrm{Prod}(\mathcal{B}^*)$ and hence $(\varinjlim \mathcal{B}, \mathrm{Prod}^+(\mathcal{B}^*))$ is a dual pair, as is $(\varinjlim \mathcal{B}, \mathrm{Prod}(\mathcal{B}^*))$.*

Proof. If $M \in \varinjlim \mathcal{B}$ then, as in the proof of 1.1, there is a pure epimorphism $\bigoplus_i B_i \to M$ with the B_i in $\mathcal{B}$, and hence a split embedding $M^* \to \prod_i B_i^*$. This proves the direction (⇒).

For the other direction, we follow [17, 4.2.18, 4.2.19]. Suppose that $M \in R$-Mod is such that $M^* \in \mathrm{Prod}(\mathcal{B}^*)$, say $i : M^* \to \prod_i B_i^*$ with $B_i \in \mathcal{B}$ is a split embedding. We may assume that the duality * is with respect to the injective cogenerator E of S-modules, where S maps to the centre of R; thus all hom groups between R-modules are S-modules.

By [4, 3.2] $\mathrm{Add}(\mathcal{B})$ is precovering so choose a precover $\bigoplus_j A_j \xrightarrow{\alpha} M$ with the $A_j \in \mathrm{Add}(\mathcal{B})$. We will show that α is surjective. Since α is a precover, for each $A \in \mathcal{B}$ the induced map $(A, \bigoplus_j A_j) \to (A, M)$ is surjective. Therefore the induced map $\mathrm{Hom}_S((A, M), E) \to \mathrm{Hom}_S((A, \bigoplus_j A_j), E)$ is injective. Since A is finitely presented we have

(e.g. [25, 25.5(ii)]) the natural isomorphism $\mathrm{Hom}_S(\mathrm{Hom}_R(A,-),E) \simeq \mathrm{Hom}_S(-,E) \otimes_R A = (-)^* \otimes_R A$, so we have that the map $M^* \otimes_R A \to (\bigoplus_j A_j)^* \otimes_R A$ induced by α is injective. These are S-modules so, by injectivity of E we have that the induced map $\mathrm{Hom}_S((\bigoplus_j A_j)^* \otimes_R A, E) \to \mathrm{Hom}_S(M^* \otimes_R A, E)$ is surjective and hence (by the Hom/$\otimes$ adjunction) that $\mathrm{Hom}_R((\bigoplus_j A_j)^*, A^*) \to \mathrm{Hom}_R(M^*, A^*)$ is surjective.

This is so for each $A \in \mathcal{B}$, in particular for each B_i, so we deduce that the map $\prod_i((\bigoplus_j A_j)^*, B_i^*) \to \prod_i(M^*, B_i^*)$ is surjective, hence that the induced map $((\bigoplus_j A_j)^*, \prod_i B_i^*) \to (M^*, \prod_i B_i^*)$ is surjective. But the latter is the map induced by $\alpha^* : M^* \to (\bigoplus_j A_j)^*$ and so we deduce that $i \in (M^*, \prod_i B_i^*)$ factors through α^* and hence, since i is monic, α^* is monic, indeed it is a pure, hence split, embedding.

Finally we use that E is a cogenerator: if α were not surjective then its cokernel would have a non-zero map to E and this, by composition with $M \to \mathrm{coker}(\alpha)$, would give a non-zero element of M^* sent to 0 by α^*, which would contradict what we have just shown.

We deduce that α is indeed an epimorphism and so the sequence $0 \to \ker(\alpha) \to \bigoplus_j A_j \xrightarrow{\alpha} M \to 0$ is exact. We have also seen that α^* is split, that is, the dual sequence $0 \to M^* \to (\bigoplus_j A_j)^* \to (\ker(\alpha))^* \to 0$ is split. Therefore the original sequence is pure, in particular α is a pure epimorphism and so (see the proof of 1.1), $M \in \varinjlim \mathcal{B}$, as required. $\square$

Examples 5.2. If $\mathcal{B} = \mathrm{add}(\mathcal{B})$ is a subcategory of R-mod then (see [1] or [3]) $\varinjlim \mathcal{B}$ is a finitely accessible category and hence is a definable category in the sense of [20], [21] but it might not be a definable *subcategory* of R-Mod: it will be closed in R-Mod under direct limits and pure submodules but it might not be closed under direct products. Indeed, the product of any collection of modules in $\varinjlim \mathcal{B}$ will have a maximal, possibly proper, submodule which lies in $\varinjlim \mathcal{B}$ and that will give the product within the category $\varinjlim \mathcal{B}$. For an illustrative example, take $\mathcal{B}$ to be the category of finite abelian groups, so $\varinjlim \mathcal{B}$ is the category of torsion abelian groups; in this case we have the almost dual pair $(\varinjlim \mathcal{B}, (\varinjlim \mathcal{B})^*)$ whose second component is the category of profinite abelian groups.

In contrast, if we take $\mathcal{B}$ to be the category of preprojective left modules over a tame hereditary algebra, then $\varinjlim \mathcal{B}$ is the category of left modules which are torsionfree in the sense of [24] (see [15, p. 743]) and $\mathcal{P}$ is the class of right modules which are divisible in the sense of that paper; each is a definable subcategory of the respective category of all modules (e.g. by [27, 3.2] and 4.3 or the above references).

If R is a finite-dimensional algebra then the classes of the form $\mathrm{Prod}((\mathcal{B} \cup \{_RR\})^*)$ for $\mathcal{B}$ an additive subcategory of R-mod are the classes of relative pure-injectives for purities determined by sets of finitely presented modules (see [16, 4.5]).

Holm says that R is "$\mathcal{B}$-coherent" if the equivalent conditions (i), (ii), (iii) of 4.3 are satisfied. The results 4.4 and 5.1 here add to this and to [7, 5.6, 5.7], at the same time removing the additional, but as it turns out unnecessary, condition $R \in \mathcal{B}$ from the latter two results.

Holm [7, 1.3] also considers the stronger condition ("$\mathcal{B}$-noetherian") that $\mathrm{Prod}(\mathcal{B}^*)$ be a definable subcategory of Mod-R. This is equivalent to the condition that every member of $(\mathrm{Prod}(\mathcal{B}^*))$ be Σ-pure-injective, where a module M is said to be **Σ-pure-injective** if $M^{(I)}$ is pure-injective for any (and then it follows, for every) infinite set I. We add, to the various characterisations [7, 1.3] of this case, the following.

Theorem 5.3. *Let $\mathcal{B}$ be an additive subcategory of R-mod. Set $M = \bigoplus\{B : B \in \mathcal{B}'\}$ where $\mathcal{B}'$ is a skeletally small version of $\mathcal{B}$. Then the following conditions are equivalent:*
(i) $\mathrm{Prod}(\mathcal{B}^*)$ is definable;
(ii) $M^* = \prod\{B^* : B \in \mathcal{B}'\}$ is Σ-pure-injective;
(iii) M is noetherian over its endomorphism ring.
In this case $\mathrm{Prod}(\mathcal{B}^*) = \mathrm{Prod}^+(\mathcal{B}^*)$ is definable and is the elementary dual of the definable subcategory $\varinjlim \mathcal{B}$.

Proof. (i)$\Rightarrow$(ii) is immediate from the fact that a definable subcategory is closed under direct sums.

(ii)$\Rightarrow$(i) It is well-known (e.g. [19, 9.34]) and easy to prove (use the result quoted in the proof of 1.1) that if a module M is Σ-pure-injective then $\mathrm{Prod}(M)$ is a definable subcategory of Mod-R.

(i)$\Leftrightarrow$(iii) It is shown in [29, Observation 8, p. 705] that if a module is (a direct summand of) a direct sum of finitely presented modules then it is noetherian over its endomorphism ring iff it has the ascending chain condition on pp-definable subgroups. That, by [29, Lemma 2, Prop. 2] (also see [20, 1.3.15]) is true of a module iff its Hom-dual has the descending chain condition on pp-definable subgroups and that, in turn, is equivalent to this Hom-dual being Σ-pure-injective (see [20, 4.4.5] for a proof and sources for this last result).

The last statement follows by 5.1. $\square$

An example here is $(R\text{-Flat}, \text{Inj-}R)$ where R is a right noetherian ring. This example illustrates that, in this situation, $\mathcal{S}$ need not equal $\mathrm{Pinj}(\mathcal{S})$. In fact, the case where both a definable category and its dual consist only of (Σ-)pure-injectives is exactly that where all the modules in these classes have finite endolength (e.g. see [20, 4.2.25 and 1.3.15]).

Proposition 5.4. *If $(\mathcal{S}, \mathcal{P})$ is an almost dual pair then the following conditions are equivalent:*
(i) both $\mathcal{P} = \mathrm{Prod}(\mathcal{S}^*)$ and $\mathcal{S} = \mathrm{Prod}(\mathcal{S}^*)$;
(ii) M^* as defined in 5.3 is of finite length over its endomorphism ring.

Example 5.5. Suppose that R is an artin algebra and that $\mathcal{B}$ is an additive subcategory of R-mod. If $\mathcal{B}$ is closed under submodules then $\varinjlim \mathcal{B}$ is definable [13, 2.2]. By 5.1 and 4.6 and the fact that * is a duality between R-mod and mod-R we deduce that if $\mathcal{B}'$ is any additive category of mod-R closed under quotients then $\mathrm{Prod}^+(\mathcal{B}')$ is a definable subcategory of Mod-R.

The pair $(\varinjlim \mathcal{B}, \mathrm{Prod}^+(\mathcal{B}^*))$ will satisfy the conditions of 5.3 iff $\mathcal{B}$ contains only finitely many indecomposable modules up to isomorphism (by [20, 4.4.31]).

If, as in Example 5.2, we take R to be a tame hereditary algebra and we take $\mathcal{P}_0$ to be the set of indecomposable preprojective left R-modules and $\mathcal{I}_0$ to be the set of indecomposable preinjective right R-modules then we have the dual pair of definable categories $(\varinjlim \mathcal{P}_0, \mathrm{Prod}^+(\mathcal{I}_0))$ (as well as $(\mathrm{Prod}^+(\mathcal{I}_0), \varinjlim \mathcal{P}_0)$) but only the second class, $\mathrm{Prod}^+(\mathcal{I}_0)$, satisfies the condition of 5.3 (unless R is of finite representation type).

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